

A CATEGORICAL CONSTRUCTION OF 4D TOPOLOGICAL QUANTUM FIELD THEORIES

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1. Introduction

In recent years, it has been discovered that invariants of three-dimensional topological objects, such as links and three-dimensional manifolds, can be constructed from various tools of mathematical physics, such as Von Neumann algebras [1], Quantum Groups [2], and Rational Conformal Field Theories [3].

Since these different structures lead to the same 3D invariants, it is natural to wonder how they are related. A fundamental connection is that they all give rise to the same special tensor categories, which act as expressions of “quantum symmetry” in the very different physical settings [4]. It is therefore very important that the 3D invariants can all be reconstructed from a tensor category with the appropriate properties, called a “modular tensor category (MTC) [3,5].” The invariants have the property that they factorize nicely if the manifold or link is cut along a surface, which we express by saying that they form a Topological Quantum Field Theory, or TQFT.

Thus the theorem proven in [3] states that a modular tensor category gives rise to a 3D TQFT. This represents a remarkable convergence with [6], in which it was realized that a suitable categorical structure would give rise to 3D topological information, although without the examples from physics.

The original suggestion to look for TQFT’s appears in (not mathematically rigorous) work of Atiyah [7] and Witten [8]. Atiyah devotes more attention to the 4D than the 3D situation, and poses the question whether the two are related; or more concretely, whether the invariants of Donaldson and Jones are related. Witten actually works in the 4D situation, and formally suggests that Donaldson’s invariant can be fitted into a 4D TQFT.

At this point, there is no mathematical construction of the 4D TQFT envisioned in [7] and [8]. Donaldson-Floer theory has so far eluded the efforts of the strongest of analysts [9].

It is therefore clear that a 4D construction of a TQFT paralleling the 3D categorical one would have very great implications for mathematics. As we shall mention in the conclusion section of this paper, there may very well be important physical implications also.

The construction we describe in this paper is a considerable step in this direction. Following a suggestion of Ooguri [10], we offer an expression which

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gives an invariant of a 4-manifold. The expression is reminiscent of the invariant of Viro and Turaev [11], and depends on a triangulation of the manifold. The relative form for a manifold with boundary, and hence the TQFT, follow directly.

We believe that the invariants, which are constructed from a modular tensor category within the 3-skeleton of the 4-manifold, are not the most general, and that the construction should be extended to a more general one, in which the MTC is replaced by a suitable 2-category.

For the purposes of this paper, all manifolds are compact, oriented, and smooth (=PL for $D=4$).

2. The Construction

In [10], Ooguri, motivated by some ideas in combinatorial quantum gravity, proposed a formal expression for an invariant for 4-manifolds. The expression he proposed was divergent, but he suggested that replacing the representations of a Lie group by those of a quantum group at a root of unity might regularize it.

Since the representations of a quantum group at a root of unity form a MTC, it is natural to recast Ooguri’s suggestion in the context of [3]. We discovered that a rather small modification of Ooguri’s expression sufficed to produce an invariant of 4-manifolds.

The expression which we found to produce a topological invariant is a sum of products of contributions for each simplex of the triangulation.

In order to define the invariant, we need to split each tetrahedron of the triangulation, so that the faces are grouped into two pairs. Geometrically, we think of this as cutting the four-holed sphere which results from thickening the 1-skeleton of the dual triangulation of the tetrahedron into two trinions (three-holed spheres). See Figure 1.

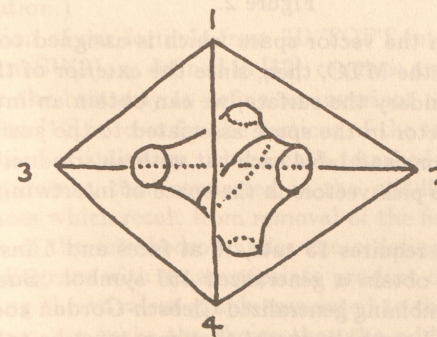


Figure 1

The surfaces which result from joining these 4-holed spheres together are very important in our definition. Since they reside in the 3-skeleton of the triangulation, we can often regard them as boundaries of 3-manifolds, so that

the theorems about invariants of manifolds with boundary in [3] apply.

We can think of a labelling of the faces of the triangulation as contributing a labeling to the holes of the 4-holed spheres. In order to pick a basis for the vector space which the MTC attaches to the four-holed sphere, we must cut it along a circle into trinions, and sum over all labelings of the internal cut by irreducible objects of the MTC. (If we change cuts, a matrix called the fusion matrix relates the two bases we obtain.)

If we glue together all the 4-holed spheres from the tetrahedra on the boundary of one 4-simplex, we obtain a surface imbedded in S^3 . See Figure 2.

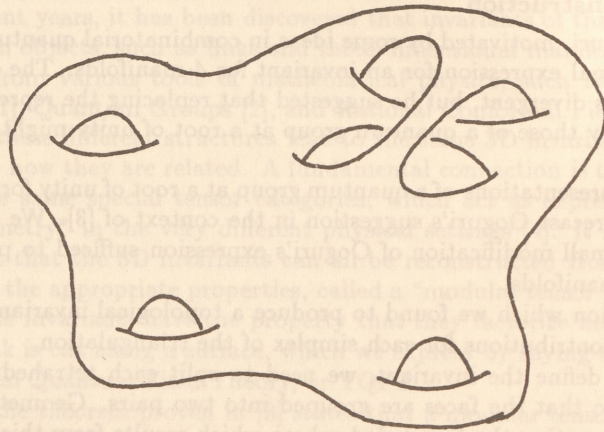


Figure 2.

If we pick a vector in the vector space which is assigned to this surface by the TQFT associated to the MTC, then, since the exterior of the surface in S^3 is a 3-manifold with boundary the surface, we can obtain an invariant number. In order to describe a vector in the space associated to the surface, we need to cut the surface into trinions and label each cut with an irreducible object. In a general MTC, we need to pick vectors in the space of intertwining operators as well.

Such a decomposition requires 15 cuts, 10 at faces and 5 inside tetrahedra. We call the invariant we obtain a generalized 15J symbol. Such an invariant can also be written by combining generalized Clebsch-Gordan coefficients, analogously to the classical 15J symbols, except that care must be taken to describe the embedding of the surface (thought of as a thickened graph) correctly, by using the braiding of the category.

Thus, for each 4-simplex, we have a set of numbers, called generalized 15J symbols, corresponding to labellings of the faces and tetrahedra of the triangula-

tion with irreducibles in the category, and a choice of two intertwining operators for each tetrahedron.

Our recipe for an invariant is as follows: for each labelling of the faces of the triangulation by an irreducible object of the category, each labelling of the cut in each tetrahedron by an irreducible, and each labelling of all the resulting trinions by intertwining operators, we take the product of all the 15J symbols of the 4-simplices of the triangulation, and correction factors corresponding to lower dimensional simplices. We then sum over all labellings.

The result is a topological invariant of the 4-manifold.

This expression is very similar to Ooguri's, except that the generalization of the 15J symbols to a MTC requires topological data, because of the braiding in the category. (As we explain below, our more technical definition of the invariant, which uses an ordering of the vertices of the triangulation, makes Ooguri's 6J symbols trivial.)

3. Proof of Topological Invariance

In the special case of quantum $SU(2)$, there are (projectively) unique intertwiners, and the various symbols admit explicit expressions in closed form [12]. In the interest of clarity, we complete the proof in that case only. We believe the generalization should work, because the proof for the case of $U_q(sl_2)$ essentially amounts to factorization on a corner for the 3D TQFT, which works for the other quantum groups as well.

In order to prove topological invariance, we need a set of moves which relate any two triangulations of the same PL manifold. We are fortunate to possess a very convenient set of moves, as a result of recent work by Pachner, in which he showed that triangulated manifolds are equivalent if and only if they are bispherically equivalent [13]. (We wish to thank W.B.R. Lickorish for bringing this to our attention.)

The translation of our formula from 3D TQFT into the recombination diagrams of quantum $SU(2)$, as defined in [12], requires a great deal of delicacy. We have found that the best way to define an invariant of 4-manifolds in terms of the expressions in [12] is to order the vertices of the triangulation and to make a consistent choice of splittings, where each tetrahedron has its odd and even faces joined separately, then connected in the middle. (By odd (even) faces, we mean the faces which result from removal of the first and third (second and fourth) vertices of the tetrahedron under the ordering). It is then necessary to normalize the diagram with an appropriate product of quantum dimensions.

In translating the trivalent graph associated to the 4-simplex into a recombination diagram, we make the choice that the odd half of each tetrahedron always comes from above. Thus, we associate to each ordered 4-simplex the following graph, or its reflection depending on whether the orientation induced

by the ordering agrees with the orientation on the 4-manifold.

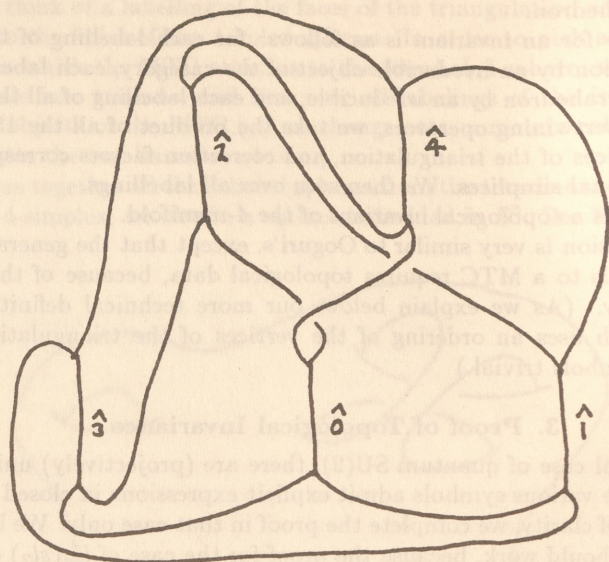


Figure 3

With this choice, Ooguri's 6J symbols become trivial, and the formula, with the proper normalization, takes the following form.

$$\sum N^{\# \text{vertices} - \# \text{edges}} \prod_{\text{faces}} \dim_q(j) \prod_{\text{tetrahedra}} \dim_q^{-1}(p) \prod_{4\text{-simplices}} 15J_q \quad (*)$$

where the sum ranges over all assignments of spins to the faces and tetrahedra of the triangulation and j represents the spin labelling a face, p represents the spin labelling the cut interior to a tetrahedron, $\dim_q$ is the quantum dimension [12], and N is the sum of the squares of the quantum dimensions.

It is now a direct matter of computation to check that our expression is invariant under Pachner's moves. Basically, the moves work because each move consists in replacing one half of the boundary of a 5-simplex by the other half.

When computing a sum of products of diagrams, in which an object in one diagram is identified with one in another diagram, we can use some simple recombination rules, which were originally discovered in the representation theory of $SU(2)$, but hold equally well for quantum $SU(2)$ (and have analogues for any

MTC). Graphically, these rules are as follows:

$$\begin{aligned} j \bigcirc &= \dim_q(j) = (-1)^{2j} [2j+1]_q = (-1)^{2j} \frac{q^{\frac{2j+1}{2}} - q^{-\frac{2j+1}{2}}}{q^{1/2} - q^{-1/2}}, \\ j \bigcup_i^{j_2} &= \frac{\sqrt{\dim_q j_2}}{\sqrt{\dim_q j}} j \bigvee_i^{j_2}, \quad j \bigcap_i^{j_2} = \frac{\sqrt{\dim_q j_1}}{\sqrt{\dim_q j}} j \bigvee_i^{j_2}, \\ j \bigcap_i^{j_2} &= \frac{\sqrt{\dim_q j_2}}{\sqrt{\dim_q j}} j \bigwedge_i^{j_2}, \quad j \bigcup_i^{j_2} = \frac{\sqrt{\dim_q j_1}}{\sqrt{\dim_q j}} j \bigwedge_i^{j_2}, \\ j \bigcap_k^l &= \delta_{jk} j \big|, \quad \sum_j j \bigwedge_k^l = k \big| \big| l, \\ j \big| \big(F \big) &= \big(F \big) j \big|, \\ j \bigvee_i^{j_2} &= (-1)^{j_1+j_2-j} q^{\frac{1}{2}(c_j - c_{j_1} - c_{j_2})} j \bigvee_i^{j_2}, \quad c_k = k(k+1), \\ j \bigvee_i^j &= \frac{1}{\sqrt{\dim_q j}} j \bigcup \end{aligned}$$

Figure 4

Notice that in the preceding figure, we have modified the convention of Kirillov and Reshetikhin, and use maxima and minima to denote multiples of the usual contraction of indices maps. Our modified normalizations are related

to those in [12] as follows

$$j \cup = (-1)^j \cup^j \text{ (Kirillov - Reshetikhin)}$$

$$j \cap = (-1)^{-j} \cap^j \text{ (Kirillov - Reshetikhin)}$$

Figure 5

Pachner's moves consist of replacing one half of the boundary of a 5-simplex with the other half. Observe that each half of the boundary of a 5-simplex is a 4-ball, and they share a common (triangulated) S^3 . The verification of invariance under Pachner's moves consists of using the diagrammatic recombination formulas of Figure 4 to show that the contribution to $(*)$ of either half of the boundary of a 5-simplex reduces to the evaluation of the invariant of labelled surfaces with trinion decompositions on the surface in the common S^3 obtained by gluing 4-holed spheres in each tetrahedron.

The computations are straightforward. The only difficulty was finding the correct generalized 15J symbol and normalizing factors so that the reductions can be carried out.

Given this description, it is probably easier for the reader to reproduce the computation than for us to write it down.

4. Implications and Extensions

Even for a MTC with a very small number of irreducibles, the number of terms in our formula is too great for paper and pencil computation. In simple cases, the use of diagrammatic formalism will allow for hand calculation. We shall endeavor to make some computer calculations in later work.

Mathematical Implications

There is a standard technique for turning an invariant based on a decomposition into tetrahedra into a TQFT [14]. Thus, we have an invariant of 4-manifolds of the same algebraic structure as the one conjectured for Donaldson-Floer theory. We are led naturally to two questions: what do our new invariants tell us about smooth 4-manifolds? and how closely are they related to Donaldson-Floer theory?

Witten [8], gives a gauge fixed form of a formal Lagrangian for Donaldson-Floer theory, which is supersymmetric. We could attempt to imitate this by

using the representations of quantum supergroups to construct a supersymmetric MTC, which might bring us closer to Donaldson-Floer theory.

It would be interesting to compute our invariants, for the MTCs at small roots of unity, on the examples of 4-manifolds which are known to be homeomorphic but not diffeomorphic [15]. This calculation would require nothing difficult, except extensive computation.

Mathematical Extensions

It is rather surprising that a 4D TQFT can be constructed from a MTC. The form of the invariant, with corrections associated to simplices of different dimensions, is suggestive of a picture in which one associates categorical structures of different levels to simplices of different dimensions. One is tempted to think that the MTC is functioning here as a 2-category with one object, or even as a 3-category with 1 object and one morphism. This would explain the most surprising aspect of Ooguri's suggestion, namely that spins appear on faces, rather than on edges, as in a gauge theory.

We are therefore led to conjecture that the invariant we calculate here is a special case of one calculated from a 2-category.

It is interesting to wonder whether the new 2-categories related to the canonical basis for quantum groups [16, 17] yield any interesting data in 4D topology.

The technique of attaching surfaces to triangulations we use is similar to the standard technique for producing a Heegaard splitting of a 3-manifold from a triangulation [22]. A Heegaard splitting can be thought of as a handlebody decomposition, in which the attaching of the 2-handles to the 1-handles is the surface map. One has the feeling that the formulation of our 4D invariant in terms of triangulations could be more elegantly restated in terms of handlebodies.

Physical Implications

The invariants of knotted graphs which are constructed from MTCs are closely related to spin networks [18, 19]. Spin networks are an old idea due to Penrose, which admit an interpretation as a discretized form of 3D quantum gravity. Furthermore, the Chern-Simons lagrangian, which Witten used to formally produce 3D TQFT [20], is also a state for quantum general relativity in the Ashtekar formalism [21]. It is a natural goal to try to relate the spin network picture to a geometric interpretation of Chern-Simons-Witten theory as a state for quantum general relativity.

The spin network picture allows us to rewrite the CSW invariant as a sum over labelings of a triangulation of a region in a 3-manifold, which can be interpreted as a sum over discretized 3-geometries [18, 21].

One is faced with the thorny problem of reinterpreting the CSW state as a 4D picture, ie, of reintroducing time into the picture. It is very suggestive to note that the construction outlined in this paper produces a 4D TQFT which is so closely related to a 3D one, and which comes from labelling a triangulation

of the 4-manifold, thus suggesting a sum over discretized geometries of the 4-manifold.

In the best of possible worlds, Einstein's equation would appear in a classical limit of a 4D summation, just as flat geometries appear in [21].²

²note added in proof: A. Ocneanu has published a note in which he argues that our invariant is a constant. We have calculated our invariant for the two levels $r=3,4$ for $S^3 \times S^1$ and for S^4 , and get different results, namely, 1 in both cases for $S^3 \times S^1$ versus 2 and 4 for S^4 . The resolution of this contradiction is that Professor Ocneanu is calculating a different formula from ours, despite a very suggestive analogy.

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Evaluating the Crane-Yetter Invariant

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I. Introduction

The purpose of this paper is to give an explicit formula for the invariant of 4-manifolds introduced by Crane and Yetter in [CR93]. For a closed 4-manifold W , this invariant will be denoted herein by $CY(W)$. Our main result is the following theorem, proved in section 4.

Theorem. Let W be a closed 4-manifold. Let $\sigma(W)$ denote the signature of W , $\chi(W)$ denote the Euler characteristic of W and $CY(W)$ denote the Crane-Yetter invariant of W . Let values N and κ be defined as in the beginning of section 4. Then $CY(W) = \kappa^{\sigma(W)} N^{\chi(W)/2}$.

This result is of general interest because it expresses the signature of a 4-manifold in terms of local combinatorial data (these data produce the state summation $CY(W)$ in terms of a triangulation of W). Our result should be compared with the work of Gelfand and Macpherson [GM92] where the Pontrjagin classes (and hence the signature) are produced by a combination of subtle combinatorics and geometric topology. Here we give a formula for the signature in terms of a topological quantum field theory that is based on $SU(2)_q$ and on q -deformed spin networks.

It should also be mentioned that the invariant $CY(W)$ is a rigorous version of ideas of Ooguri [OG92]. It is of interest to examine the implications of